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Vortices in the Landau–Ginzburg model of the quantized Hall effect

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Abstract. The 'Landau–Ginzburg' theory of Girvin and MacDonald, modified by adding the natural magnetic term, is shown to admit stable topological as well as non-topological vortex solutions. The system is the common $\lambda \to 0$ limit of two slightly different non-relativistic Maxwell–Chern–Simons models of the type recently introduced by Manton. The equivalence with the model of Zhang, Hansson and Kivelson is demonstrated.

In [GIR], Girvin and MacDonald presented a 'Landau–Ginzburg' theory for the quantum Hall effect (QHE). On phenomenological grounds, they suggest representing the off-diagonal long-range order (ODLRO) by a scalar field $\psi(\boldsymbol{x})$ on the plane, and that the frustration due to deviations away from the quantized Laughlin density is caused by an effective gauge potential $\boldsymbol{a}(\boldsymbol{x})$. We propose to describe this static planar system by the Lagrange density

$$\mathcal{L} = \tfrac{1}{2}b^2 + |\boldsymbol{D}\psi|^2 + \mathrm{i}\phi(|\psi|^2 - 1) - \mathrm{i}\frac{\kappa}{2}(\phi\boldsymbol{\nabla} \times \boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{\nabla}\phi) \tag{1}$$

where $b = \boldsymbol{\nabla} \times \boldsymbol{a}$ is the effective magnetic field, $\boldsymbol{D} = \boldsymbol{\nabla} + \mathrm{i}\boldsymbol{a}$ is the gauge-covariant derivative, and the Lagrange multiplier ϕ is a scalar potential. Equation (1) only differs from the original expression of Girvin and MacDonald in our having added the natural magnetic term, $b^2/2$, also present in conventional Landau–Ginzburg theory [LP]. The associated equations of motion read

$$\boldsymbol{D}^2\psi = \mathrm{i}\phi\psi \tag{2a}$$

$$\kappa b = |\psi|^2 - 1 \tag{2b}$$

$$\boldsymbol{\nabla} \times b - \mathrm{i}\kappa\boldsymbol{\nabla} \times \phi = -\boldsymbol{\jmath} \tag{2c}$$

where $\boldsymbol{\jmath} = -\mathrm{i}(\psi^*\boldsymbol{D}\psi - \psi(\boldsymbol{D}\psi)^*)$ the current. The first is a static, gauged Schrödinger equation. The second is the relation proposed by Girvin and MacDonald to relate the magnetic field to the particle density. Note here the -1 coming from the weird term $-\mathrm{i}\phi$ in the Lagrangian, and representing the background charge [MAN]. The last equation is the Ampère–Hall law: $\boldsymbol{e} = -\mathrm{i}\boldsymbol{\nabla}\phi$ is an effective electric field, so that κ is interpreted as the *Hall conductance*.

This system is rather similar to those studied in Chern–Simons field theory [JP], and in particular to that recently introduced by Manton [MAN, HHY]. Using these techniques,

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(i) we construct, with the method of Bogomolny [BOG], stable vortex solutions; (ii) point out that (1) is the $\lambda \to 0$ limit of two slightly different systems; (iii) demonstrate the equivalence with another 'Landau-Ginzburg' model, introduced by Zhang *et al* [ZH].

Let us try to reduce the second-order equations to the first-order 'self-dual' system

$$(D_1 \pm \mathrm{i}D_2)\psi = 0 \qquad \kappa b = |\psi|^2 - 1. \tag{3}$$

From the first of these relations we infer that $\boldsymbol{D}^2 = \pm b$ and $\boldsymbol{\jmath} = \mp \boldsymbol{\nabla} \times \varrho$. Inserting this into the Schrödinger equation determines the multiplier field as $\phi = \mp\frac{\mathrm{i}}{\kappa}(\varrho - 1)$. It follows that, from Ampère's law, the Hall conductance κ has to be

$$\kappa = \pm\tfrac{1}{2}. \tag{4}$$

The vector potential is expressed using the self-dual ansatz (3) as $\boldsymbol{a} = \mp\frac{1}{2}\boldsymbol{\nabla} \times \log\varrho + \boldsymbol{\nabla}\omega$, where ω is an arbitrary real function chosen so that $\boldsymbol{a}$ is regular [JP]. Inserting this into (2*b*) we obtain, for both signs, the 'Liouville-type' equation

$$\Delta \log \varrho = 4(\rho - 1). \tag{5}$$

Note that any solution will carry an effective magnetic as well as an electric field.

Now we have to find what kind of solutions we are interested in. To do this, let us consider the energy,

$$H = \int \left\{ \frac{b^2}{2} + |\boldsymbol{D}\psi|^2 \right\} \mathrm{d}^2\boldsymbol{x}. \tag{6}$$

Using the identity $|\boldsymbol{D}\psi|^2 = |(D_1 \pm \mathrm{i}D_2)\psi|^2 \mp b|\psi|^2 +$ (surface term), as well as (2*b*), the energy is rewritten as

$$H = \int \left\{ |(D_1 \pm \mathrm{i}D_2)\psi|^2 + \left(\frac{1}{2\kappa^2} \mp \frac{1}{\kappa}\right)|\psi|^4 + \frac{1}{2\kappa^2} + \left(-\frac{1}{\kappa^2} \pm \frac{1}{\kappa}\right)|\psi|^2 \right\} \mathrm{d}^2\boldsymbol{x}.$$

The quartic term disappears when $\kappa = \pm\frac{1}{2}$, leaving us with

$$H = \int \{|(D_1 \pm \mathrm{i}D_2)\psi|^2 + 2(1 - |\psi|^2)\}\, \mathrm{d}^2\boldsymbol{x}. \tag{7}$$

The system admits the zero-energy ground state (condensate) $\psi \equiv 1$ and $\boldsymbol{e} = 0$, $b = 0$.

Then we have two possibilities.

• Either, to get finite energy, we can require that at infinity the solution tends to the condensate state, $|\psi| \to 1$ and $|\boldsymbol{D}\psi| \to 0$. These two conditions imply that the magnetic flux is quantized,

$$\Phi \equiv \int b\, \mathrm{d}^2x = 2\pi n \tag{8}$$

where the integer n is the 'winding number' of ψ, which maps the circle at infinity into $U(1)$. Then, using equation (2*b*), the 'particle number'

$$N = \int (1 - |\psi|^2)\, \mathrm{d}^2\boldsymbol{x} \tag{9}$$

is finite, and is related to the flux as $N = -\kappa\Phi = -2\pi\kappa n$. ($N$ is conserved owing to the continuity equation which follows from equation (2*a*).) Our finite-energy solutions (referred to as *topological vortices*) therefore carry a non-vanishing flux as well as a charge. Their energy satisfies, by equation (7), the 'Bogomolny' inequality

$$H \geqslant 2N = -4\pi\kappa n = \mp 2\pi n. \tag{10}$$

To get a positive bound, κ and n must have opposite signs: the upper (lower) sign works for $n < 0$ (resp. for $n > 0$).

The 'Bogomolny' bound (10) is saturated when the self-duality equations (3) hold. (This is in fact the case of 'Bogomolny' vortices in the Abelian Higgs model, so that equation (5) admits a $2n$-parameter family of solutions [WEIN].) Since they correspond to the absolute minima of the energy, such solutions are stable.

• Another possibility is, however, to choose $b = b_0 = -1/\kappa$, $\boldsymbol{e} = 0$, $\psi \equiv 0$ as the ground state. This is *not* a finite-energy solution, though subtracting its (constant) energy density $1/2\kappa^2 = 2$ from (7), we get

$$\overline{H} = \int \left\{ \frac{b^2}{2} - \frac{b_0^2}{2} + |\boldsymbol{D}\psi|^2 \right\} \mathrm{d}^2x = \int \{ |(D_1 \pm \mathrm{i}D_2)\psi|^2 - 2|\psi|^2 \} \, \mathrm{d}^2x. \quad (7')$$

Now we can look for *non-topological* solutions, i.e. whose such 'number', defined as

$$\overline{N} = \int |\psi|^2 \, \mathrm{d}^2x \quad (9')$$

converges. This 'renormalized number' $\overline{N}$ is now a continuous, rather than a quantized parameter, which takes any positive value. Then we get the modified Bogomolny bound

$$\overline{H} \geqslant -2\overline{N} \quad (10')$$

with the equality attained when the self-duality equations (3) hold. (Remember that $\overline{H}$ is the *relative* energy with respect to the infinite-energy background.)

Since $\psi \to 0$ at infinity, the condition $D\psi \to 0$ does not now imply a quantized flux. The integral (8) is indeed infinite, as one sees directly in the radial case from equation (12) below. However, subtracting the constant background magnetic field, $b = b_0 = -1/\kappa$, we get the renormalized number $\overline{N}$ in equation (9′):

$$\overline{\Phi} \equiv \int \overline{b} \, \mathrm{d}^2x \equiv \int (b - b_0) \, \mathrm{d}^2x = \int \frac{1}{\kappa} |\psi|^2 \, \mathrm{d}^2x = \frac{1}{\kappa}\overline{N}. \quad (8')$$

This 'renormalized flux' therefore depends on a continuous parameter, namely on $\overline{N}$, just as for relativistic non-topological solitons [JLW].

The most convenient way of studying the solutions is to work directly with the *first-order equations* (3) rather than with the second-order equation (5). Assuming that the fields have the form $\psi = f(r)\mathrm{e}^{\mathrm{i}n\theta}$, $a_r = 0$, $a_\theta = a(r)$, the self-dual (SD) equations read

$$f' = \pm \frac{n + a}{r} f \qquad \frac{a'}{r} = \pm 2(f^2 - 1). \quad (11)$$

Regularity at the origin requires n and κ to be correlated as sign $n = -$sign κ, so that we get *chiral solitons*. The small-r behaviour is $f(r) = \alpha r^{|n|}$, $a = \mp r^2$, where α is a real parameter. For large r, we find instead

$$\left.\begin{array}{l} f(r) \sim 1 - C K_0(2r) \sim 1 - C \frac{\mathrm{e}^{-2r}}{\sqrt{r}} \\ b \sim D r K_1(2r) \sim D \frac{\mathrm{e}^{-2r}}{\sqrt{r}} \end{array}\right\} \text{ for a topological vortex}$$
$$\left.\begin{array}{l} f(r) \sim \mathrm{e}^{-r^2} \\ b \sim \mp 2(1 - \mathrm{e}^{-2r^2}) \end{array}\right\} \text{ for a non-topological vortex.} \quad (12)$$

A simple numerical calculation shows that, for each integer value of n, there is just one radially symmetric topological vortex obtained for $\alpha = \alpha_0(n)$, while non-topological vortices arise for an entire range $\alpha < \alpha_0(n)$ of the parameter. This behaviour is understood by looking, following Ezawa *et al* [EZA], at the second-order equation (5). Again, restricting

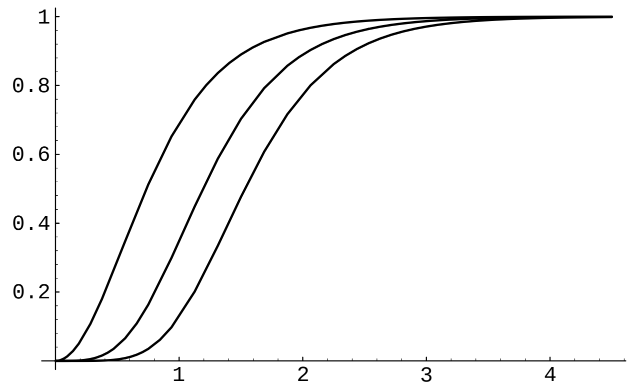

Figure 1. The order parameter density of the radially symmetric topological vortices with winding number $n = 1, 2, 3$. For each value of n, there is exactly one vortex.

ourselves to the radial case, we can view $x = \log f$ as the 'position' and the radial coordinate, $t \equiv r > 0$, as 'time', so that equation (5) becomes

$$\ddot{x} = -\frac{1}{t}\dot{x} - \boldsymbol{\nabla} U \qquad U(x) \equiv (2x - \mathrm{e}^{2x}). \tag{13}$$

This is the equation of motion of a classical particle in a time-dependent frictional force and an external potential $U(x)$. Observe that $U(x)$ increases from $x = -\infty$, reaches its maximum at $x = 0$, and then decreases. As $t \to \infty$, $f \to 1$ i.e. $x \to 0$ for a topological soliton, and $f \to 0$ i.e. $x \to -\infty$ for a non-topological soliton. The regularity of ψ requires f to vanish at the origin. Let us therefore consider a 'particle' which starts at 'time' $t = 0$ from the 'position' $x = -\infty$. If its 'energy' is not sufficient to climb the potential hill, it will, after some time, fall back to $x = -\infty$ as $t \to \infty$: we get a non-topological soliton. Increasing the initial velocity, we can make our particle approach $x = 0$ when $t \to \infty$, yielding a topological soliton. Clearly, this only happens for some specific initial 'energy', corresponding to a specific value $\alpha = \alpha_0(n)$ of the initial parameter. If the 'energy' is even higher, the particle overshoots: the required boundary conditions cannot be satisfied, so that no self-dual vortex can exist.

The appearance of these two different types of vortex solutions can be understood by adding a self-interaction potential, $U(\psi)$, to the Lagrangian. The use of such a potential is quite common in Landau–Ginzburg theory (see [LP, p 179]). In the context of the QHE, it can be viewed as the remnant of the two-body potential in the second-quantized Hamiltonian for spin-polarized electrons, when the effective theory is derived [ZH].

- For the symmetry breaking potential

$$U(\psi) = \frac{\lambda}{8}(1 - |\psi|^2)^2 \tag{14}$$

finite energy requires $|\psi| \to 1$ at infinity. The Bogomolny equation (3) then yields topological vortex solutions, provided

$$\lambda = -\frac{4}{\kappa^2} \pm \frac{8}{\kappa}. \tag{15}$$

The potential in (14) is physically admissible (repulsive) when $\lambda \geqslant 0$, i.e. $|\kappa| \geqslant \frac{1}{2}$.

- For the non-symmetry-breaking potential

$$\overline{U} = C + \frac{\lambda}{8}|\psi|^4 \tag{16}$$

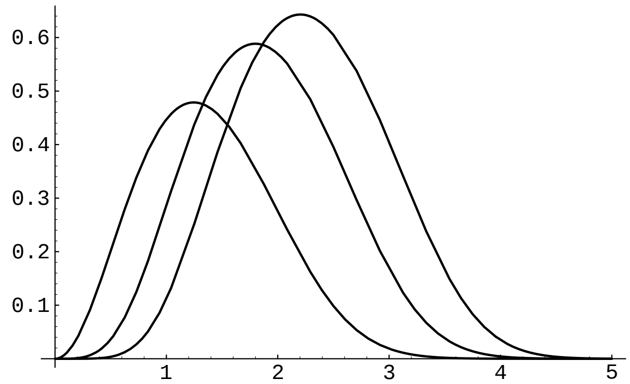

Figure 2. The order parameter density of the radially symmetric non-topological vortices for $n = 1, 2, 3$. For each value of n, there is a full range of vortices.

finite energy requires instead $|\psi| \to 0$ at infinity, and the Bogomolny trick works when $C = -\frac{1}{2}\kappa^2$ and for λ as in equation (15). The potential (16) is physically admissible (attractive) when $\lambda \leqslant 0$, i.e. when $|\kappa| \leqslant \frac{1}{2}$. Changing λ from a negative to a positive value can be viewed as a phase transition, with transition point $\lambda = 0$.

Let us stress that there is no way to obtain the quantization of the Hall conductance from our classical field theory: the condition (4) is merely replaced by equation (15).

Thus, the modified Girvin model (1) can be viewed as the $\lambda \to 0$ limit of *two*, slightly different systems, one of them correct for $\kappa \geqslant \frac{1}{2}$, the other for $\kappa \leqslant \frac{1}{2}$. In both cases, there is a natural boundary condition at infinity, dictated by finite energy. However, $|\kappa| \to \frac{1}{2}$ when $\lambda \to 0$, so that both conditions can be used. In this limit, the boundary condition at infinity has to be put in manually, since one cannot know what stays 'behind' the coefficient $\lambda = 0$. (This also happens for 't Hooft–Polyakov monopoles in the Prasad–Sommerfeld limit.)

In [HHY], we show that, for potential (14) with $\lambda \neq 0$, the system admits a six-parameter group of symmetry, made of the unbroken parts of the 'geometric' and 'hidden' Schrödinger symmetries. It is easy to see that, for $\lambda = 0$ some of the obstructions are lifted so that 'imported' dilatations and expansions are also unbroken. Thus, we have the full 'hidden' Schrödinger symmetry, just like for the purely quartic potential [EZA, JPH].

It is worth pointing out that the field-theoretical generalization of the Girvin model is the *non-relativistic Maxwell–Chern–Simons* system proposed by Manton [MAN], whose $(2+1)$-dimensional Lagrangian reads

$$\frac{1}{2}b^2 - \frac{\mathrm{i}}{2}(\psi^* D_t\psi - \psi(D_t\psi)^*) + |\boldsymbol{D}\psi|^2 + U(\psi) - \frac{\kappa}{2}(ba_t + \boldsymbol{a}\times\boldsymbol{e}) + a_t + \boldsymbol{a}\cdot\boldsymbol{\jmath}^T \tag{17}$$

where the constant vector $\boldsymbol{\jmath}^T$ (called the transport current) has been included for the sake of Galilean invariance. Note also the absence of an electric term. In a suitable Galilei frame $\boldsymbol{\jmath}^T$ vanishes [MAN] and, for $U(\psi) \equiv 0$ and identifying a_t with $-\mathrm{i}\phi$, the modified Girvin system (1) is recovered, when time independence is assumed.

So far, we have been working with a spinless matter field. Introducing spin would not change the situation, though. Modifying the Lagrangian (1) as

$$\mathcal{L}_{\text{spin}} = \tfrac{1}{2}b^2 + (\boldsymbol{D}\Psi)^\dagger(\boldsymbol{D}\Psi) + \mathrm{i}\phi\big(\Psi^\dagger\Psi - 1\big) - b\Psi^\dagger\sigma_3\Psi - \mathrm{i}\frac{\kappa}{2}(b\phi + \boldsymbol{a}\times\boldsymbol{\nabla}\phi) \tag{1$''$}$$

where Ψ is a two-component Pauli spinor, would replace (2a) by the Pauli equation

$$\mathrm{i}\phi\Psi = [\boldsymbol{D}^2 + b\sigma_3]\Psi \tag{2a''}$$

while (2*b*) and (2*c*) remain unchanged up to $\boldsymbol{\jmath} = -\mathrm{i}(\Psi^\dagger \boldsymbol{D}\Psi - (\boldsymbol{D}\Psi)^\dagger\Psi) + \boldsymbol{\nabla}\times(\Psi^\dagger\sigma_3\Psi)$. Then, for $\kappa = \pm(\frac{1}{2})$, the spinning system admits self-dual solutions of definite chirality, $\Psi_+ = \binom{0}{\psi_+}$ and $\Psi_- = \binom{\psi_-}{0}$, with $\varrho = |\Psi_\pm|$ satisfying the *same* Liouville-type equation (5) [HHY].

In [ZH], Zhang *et al* proposed another 'Landau–Ginzburg' theory for the QHE. They consider a scalar field ψ coupled to a gauge field A_μ, described by the Lagrangian

$$\mathcal{L}_Z = 4\theta\epsilon^{ij}(2A_0\partial_i A_j - A_i\partial_0 A_j) - \frac{1}{4\theta}\epsilon^{\mu\nu\sigma}A_\mu\partial_\nu A_\sigma$$
$$+\psi^*[\mathrm{i}\partial_0 - (A_0 + A_0^{\mathrm{ext}})]\psi + \psi^*[-\mathrm{i}\boldsymbol{\nabla} - (\boldsymbol{A}+\boldsymbol{A}^{\mathrm{ext}})]^2\psi + U(\psi) \qquad (18)$$

where A_μ^{ext} is the vector potential of an external electromagnetic field, and $U(\psi) = \mu|\psi|^2 - \lambda|\psi|^4$ is a quartic self-interaction potential. They argue that their theory is different from that of Girvin and MacDonald. We show, however, that for a static system in a purely magnetic background and for $U(\psi) \equiv 0$, the two models are indeed *mathematically equivalent*. To see this, let us note that under the above restrictions, after some partial integrations and dropping surface terms, the Lagrangian of Zhang *et al* becomes

$$\left(4\theta - \frac{1}{4\theta}\right)\epsilon^{\mu\nu\sigma}A_\mu\partial_\nu A_\sigma - A_0|\psi|^2 + |(-\mathrm{i}\boldsymbol{\nabla} - (\boldsymbol{A}+\boldsymbol{A}^{\mathrm{ext}})\psi|^2. \qquad (19)$$

On the other hand, the Girvin–MacDonald model can also be presented in a slightly different way. Let us indeed consider a static, purely magnetic external field B^{ext}. Then, setting $\boldsymbol{a} = -\boldsymbol{A} - \boldsymbol{A}^{\mathrm{ext}}$ and $A_0 = -\mathrm{i}\phi$, we find that, for the choice

$$\kappa = \frac{1}{B^{\mathrm{ext}}} \qquad (20)$$

the (original) Girvin–MacDonald Lagrangian (i.e. (1) *without* the $b^2/2$ term) becomes, up to a surface term,

$$\mathcal{L}_G = |\boldsymbol{\nabla} - \mathrm{i}(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{ext}})\psi|^2 - A_0|\psi|^2 - \frac{\kappa}{2}(A_0\boldsymbol{\nabla}\times\boldsymbol{A} + \boldsymbol{A}\times\boldsymbol{\nabla}A_0) \qquad (21)$$

which is indeed (19) when $\kappa = -8\theta + \frac{1}{2}\theta$†.

In [EZA] Ezawa *et al* have shown that the model of Zhang *et al* admits, for a suitable choice of the self-interaction potential $U(\psi)$, topological as well as non-topological vortex solutions. In the light of our results we see that, alternatively, we can add a b^2 term to the Lagrangian while still working with $U(\psi) = 0$.

As to the physical significance of our solutions, our 'topological' vortices correspond to quasiparticles and quasiholes. The physical interpretation of our 'non-topological' vortices is, however, not yet clear. We are currently working on this issue.

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† In the reprint volume of M Stone (1992 *Quantum Hall Effect* (Singapore: World Scientific)), the first term in equation (17) has been suppressed. This does not change the conclusion: the two models are still equivalent when $\kappa = \frac{1}{2}\theta$.

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